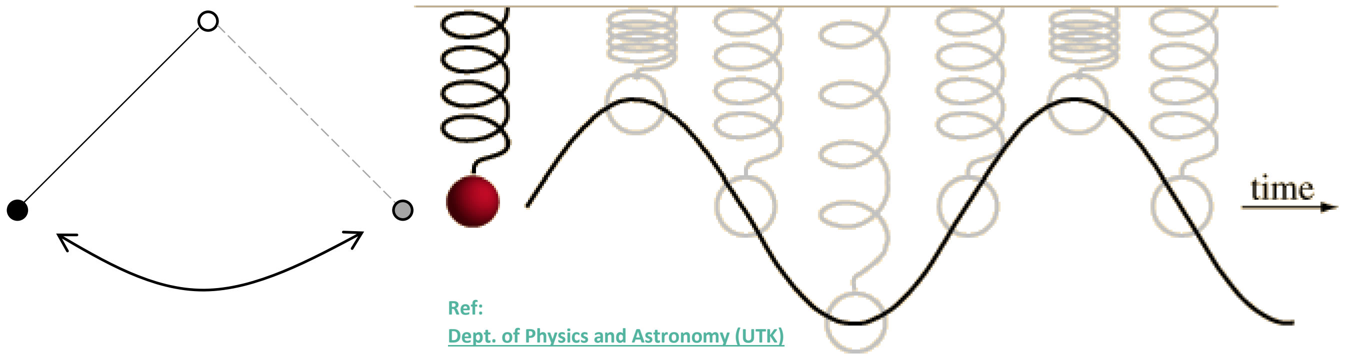
Differential Equations IV Cheat Sheet (A Level Only)

Introduction to Simple Harmonic Motion

Simple harmonic motion describes systems involving regular oscillations. Some classic examples include a pendulum moving back and forth or a spring bouncing up and down. This is illustrated in the diagrams below.



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Simple harmonic motion is characterised by the following differential equation: $\ddot{x} = -\omega^2 x$, where x is the displacement of an object and ω is its angular frequency, which is closely related to the frequency of oscillation as $\omega = 2\pi f$. Observe the notation $\ddot{x} = \frac{d^2x}{dt^2}$ which is also known as the acceleration. The maximum displacement of the object is known as the amplitude and the time for a full oscillation is described as the period. Furthermore, the equilibrium position the object oscillates about is referred to as the central line. Notice that the motion is symmetrical about the central line and the object is always accelerating towards it.

Solving the Equation for Simple Harmonic Motion

The simple harmonic motion equation is of a form already covered in the Differential Equations II Cheat Sheet, using the auxiliary equation. It has complex roots which results in the general solution containing sinusoidal waves - to be expected from the oscillatory nature of the motion. The solution to the equation for simple harmonic motion equation is shown below.

Write equation in the standard form.	$\ddot{x} + \omega^2 x = 0$
Write and solve auxiliary equation.	$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega$
Write general solution.	$x = A\cos\omega t + B\sin\omega t$

Often when modelling a system with simple harmonic motion, at $t = 0$ the object either starts on the central line or at the maximum displacement from the central line. If the amplitude is a then the solution in these cases is $x = a\sin\omega t$ and $x = a\cos\omega t$, respectively. Alternatively, using trigonometry, the general solution can be rewritten as $x = R\sin(\omega t + \epsilon)$ where R is amplitude and ϵ is the phase shift. Another useful result is a relationship between the displacement of an object and its velocity. This relationship is: $v^2 = \omega^2(a^2 - x^2)$ where v, ω, a, x are the velocity, angular frequency, amplitude, and displacement respectively. The derivation is shown below.

Assume the object begins at the central line (this relationship applies to all simple harmonic motion, just choose to start time here for convenience).	$x = a\sin\omega t$
Differentiate with respect to time to obtain the velocity.	$v = \dot{x} = a\omega\cos\omega t$
Square both sides and use a trigonometric identity to simplify. Recall $x = a\sin\omega t$ so $x^2 = a^2\sin^2\omega t$.	$v^2 = a^2\omega^2\cos^2\omega t \Rightarrow v^2 = a^2\omega^2(1 - \sin^2\omega t) = \omega^2(a^2 - x^2)$

Example 1: At $t = 0$, a particle attached to the end of a spring is pulled down to oscillate in simple harmonic motion with an amplitude of 0.13m. As the particle crosses the central line it has a speed of 4ms^{-1}.

a) Calculate the time period of the oscillation
b) Find the speed of the particle when $t = 5s$

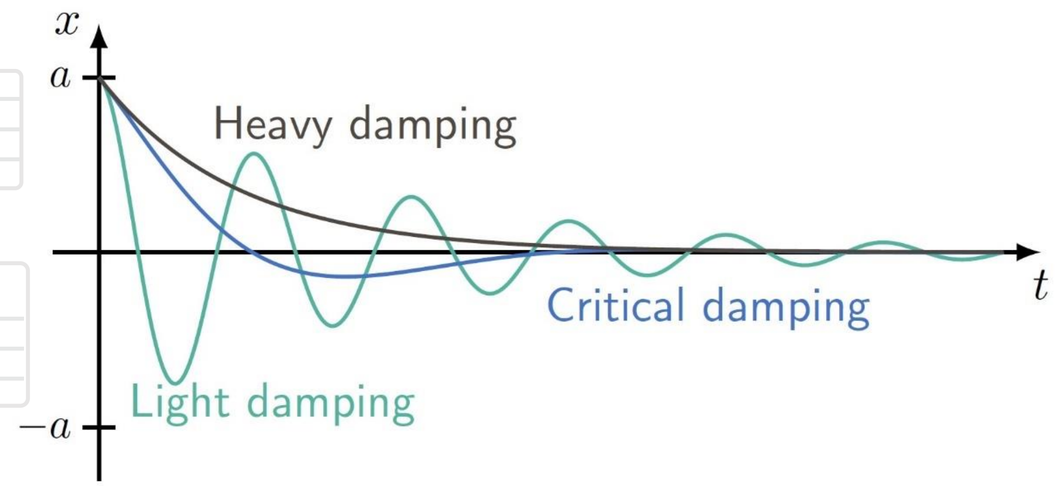
a) Substitute known values into the equation $v^2 = \omega^2(a^2 - x^2)$ and rearrange to find ω. Recall the sign of ω describes the direction of motion so does not affect the time period, so take sign to be positive.	$\omega = \sqrt{\frac{4^2}{0.13^2 - 0^2}} = \frac{400}{13} \Rightarrow \text{T} = \frac{2\pi}{\omega} = \frac{13\pi}{200} = 0.204\text{s (3s.f.)}$
b) Use $x = a\cos\omega t$ as particle begins at the maximum displacement and differentiate to obtain v.	$v = \dot{x} = -a\omega\sin\omega t$
Take the modulus as speed is the magnitude of velocity and substitute values in. Recall all angles are in radians.	$\lvert v\rvert = (0.13)\left(\frac{4}{0.13}\right)\sin\left(\left(\frac{4}{0.13}\right)(5)\right) = 4\sin\frac{2000}{13} = 0.367\text{ms}^{-1}\text{ (3s.f.)}$

Modelling Damped Harmonic Motion

With the current model motion will continue indefinitely. Obviously, this is not realistic in the real world due to frictional forces acting on the object. This model can be adapted to include a drag force D which is modelled as proportional to the velocity but acting in the opposite direction to motion, so $D = -Kv$. After incorporating this into the previous model, the following differential equation is obtained: $\ddot{x} + k\dot{x} + \omega^2 x = 0$, where $k = \frac{K}{m}$. To solve this 2nd order homogenous differential equation with constant coefficients, it is standard to divide through by the coefficient of $\ddot{x}$ which in this case would have been the mass m. As discussed on Differential Equations II Cheat Sheet, these differential equations have a general solution of 3 different forms depending on the discriminant of the auxiliary equation. In this context they are given names as shown in the diagram below.

Type of damping	Discriminant
Light	$\Delta = k^2 - 4\omega^2 > 0$
Critical	$\Delta = k^2 - 4\omega^2 = 0$
Heavy	$\Delta = k^2 - 4\omega^2 < 0$

Type of Damping	Analytic Form of Solution
Light	$x = e^{\alpha t}(a\cos\beta t + b\sin\beta t)$
Critical	$x = (A + Bt)e^{\lambda t}$
Heavy	$x = Ae^{\lambda_1} + Be^{\lambda_2}$



Example 2: A particle is connected to a spring which is submerged in oil such that as the spring oscillates the particle is subjected to a drag force which is modelled to be proportional to the speed of the particle acting in the opposite direction to the particle's motion. The displacement of the particle is modelled by the differential equation $\ddot{x} + k\dot{x} + 9x = 0$. For what value of k is the particle critically damped?

Write the auxiliary equation of the differential equation.	$\lambda^2 + k\lambda + 9 = 0$
Consider the discriminant, for critical damping $\Delta = 0$. Solve for k, noticing that from the context k must be positive.	$\Delta = k^2 - 36 = 0 \Rightarrow k = 6$

Example 3: A child is pushed on a swing. The child's angular displacement θ from the original equilibrium position is modelled by the differential equation:

$$\ddot{\theta} + 2\dot{\theta} + 26\theta = 0$$

a) Find the particular solution of this differential equation if initially the child is pushed from the equilibrium position with angular speed of $\frac{5\pi}{2}$ rad s^{-1}.

b) Hence sketch a graph of the subsequent motion of the child and classify the type of damping.

a) Write and solve the auxiliary equation of the differential equation.	$\lambda^2 + 2\lambda + 26 = 0 \Rightarrow \lambda = -1 \pm 5i$
Write general solution.	$\theta = e^{-t}(A\cos 5t + B\sin 5t)$
Use initial position to fix A.	Initially child at equilibrium position $\Rightarrow$ A $= 0$
Differentiate to use initial velocity to fix B.	$\dot{\theta} = e^{-t}(5B\cos 5t - B\sin 5t) \Rightarrow \frac{5\pi}{2} = e^{-(0)}(5B\cos 0 - B\sin 0) \Rightarrow \text{B} = \frac{\pi}{2}$
Write the particular solution.	$\theta = \frac{\pi}{2}e^{-t}\sin 5t$
b) Sketch the graph. From its shape or from the discriminant classify the damping as light.